2

Integration

Structure

2.1. Introduction. This chapter contains results related to finding the integration of a given function which help students in further studies of curves in various fields.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Integration.
(ii) Definite integrals.
(iii) Finding area.
(iv) Leontiff Input-Output Model.

2.1.2. Keywords. Integrate, Model, Definite Integral.

2.2. Integration. We will consider the inverse process of differentiation. In differentiation, we find the differential co-efficient of a given function while in integration if we are given the differential co-efficient of a function, we have to find the function. That is why integration is called anti-derivative i.e. in differentiation if $y = f(x)$ we find $\frac{dy}{dx}$. In integration, we are given $\frac{dy}{dx}$ and we have to find y. This integration is also called indefinite integral.

2.2.1. Definition of Integration

Integration is the inverse process of differentiation.

If $\frac{d}{dx}[\varphi(x)] = f(x)$ then

$\varphi(x)$ is called the integral or anti-derivative or primitive of $f(x)$ with respect to x.

Symbolically, it is written as

$$\int f(x)dx = \varphi(x)$$

The symbol $\int \quad dx$ denotes integration w.r.t. x. Here dx conveys that x is a variable of integration. The given function whose integral is to be found, is known as integrand.

2.2.2. Example. $\frac{d}{dx}(x^2) = 2x$

$$\therefore \int 2x \, dx = x^2.$$

2.2.3. Constant of integration

We know that $\frac{d}{dx}(x^3) = 3x^2$

Therefore integral of $3x^2$ may be $x^3, x^3 + 1$ or $x^3 + C$ where C is any arbitrary constant. Thus

$$\int 3x^2 \, dx = x^3 + C$$

2.2.4.Example. Find $\int 5x^6 \, dx$

Solution. $\int 5x^6 \, dx = 5\int x^6 \, dx = 5 \times \frac{x^7}{7} + C = \frac{5}{7}x^7 + C$

2.2.5. Standard Formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \ n \neq -1 \quad \left[since \ \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n\right]$
2. $\int \frac{1}{x} dx = log_e x + C \quad \left[since \ \frac{d}{dx}(log_e x) = \frac{1}{x}\right]$
3. $\int e^x \, dx = e^x + C, \quad \left[since \ \frac{d}{dx}(e^x) = e^x\right]$
4. $\int a^x \, dx = \frac{a^x}{log_e a} + C, \quad \left[since \ \frac{d}{dx}\left(\frac{a^x}{log_e a}\right) = a^x\right]$
5. $\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + C, \quad \left[since \ \frac{d}{dx}(e^{ax+b}) = ae^{ax+b}\right]$

6. $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,$ $\left[since \ \frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{a(n+1)}\right) = (ax+b)^n\right]$ $(if \ n \neq -1)$

7. $\int \frac{dx}{ax+b} dx = \frac{1}{a} \log|ax+b| + C$ $\left[since \ \frac{d}{dx}\left(\frac{\log|ax+b|}{a}\right) = \frac{1}{ax+b}\right]$

2.2.6. Theorem. The integral of the product of a constant and a function is equal to the product of a constant, and integral of the function i.e., $\int kf(x)dx = k\int f(x)dx,$ k being a constant.

Proof. Let $\int f(x)dx = \varphi(x)$, $\therefore \frac{d}{dx}[\varphi(x)] = f(x)$

Now $\frac{d}{dx}[k\varphi(x)] = k.\frac{d}{dx}[\varphi(x)]$

[Since, the derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function]

$$= kf(x) \qquad \left(as \ \frac{d}{dx}[\varphi(x)] = f(x)\right)$$

Thus, by definition

$$\int k.f(x)dx = k.\varphi(x) = k.\int f(x)dx.$$

2.2.7. Theorem. The integral of the sum or the difference of two functions is equal to the sum or difference of their integrals i.e., $\int[f_1(x) \pm f_2(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx.$

Proof. Let $\int f_1(x)dx = \varphi_1(x)$ and $\int f_2(x)dx = \varphi_2(x)$

Therefore,

$$\frac{d}{dx}[\varphi_1(x)] = f_1(x) \text{ and } \frac{d}{dx}[\varphi_2(x)] = f_2(x)$$

Now $\frac{d}{dx}[\varphi_1(x) \pm \varphi_2(x)] = \frac{d}{dx}[\varphi_1(x)] \pm \frac{d}{dx}[\varphi_2(x)] = f_1(x) \pm f_2(x)$

[Since, the derivative of the sum or difference of two functions is equal to the sum or difference of their derivatives].

Therefore, by definition of the integral of a function

$$\int[f_1(x) \pm f_2(x)]dx = \varphi_1(x) \pm \varphi_2(x) = \int f_1(x)dx \pm \int f_2(x)dx.$$

Remark. We can extend this theorem to a finite number of functions and can have the following result.

$$\int[f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \cdots \pm \int f_n(x)dx.$$

2.2.8. Example. Write down the integral of

(i) x^2 (ii) x^{-9} (iii) 1

(iv) $\sqrt{x}$ (v) $\frac{1}{x^2}$ (vi) $x^{-2/3}$

Solution.

(i) $\int x^2 dx = \frac{x^{2+1}}{2+1} + C = \frac{1}{3}x^3 + C$

(ii) $\int x^{-9}dx = \frac{x^{-9+1}}{-9+1} + C = \frac{1}{-8}x^{-8} + C = -\frac{1}{8x^8} + C$

(iii) $\int 1dx = \int x^0 dx = \frac{x^{0+1}}{0+1} + C = x + C$

(iv) $\int x^{1/2}dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{2}{3}x^{3/2} + C$

(v) $\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} + C = -\frac{1}{x} + C$

(vi) $\int x^{-2/3}dx = \frac{x^{-2/3+1}}{-2/3+1} + C = 3x^{1/3} + C.$

2.2.9. Example. Find the integrals of the following

(i) $\sqrt{x} - \frac{1}{\sqrt{x}}$ (ii) $\frac{(1+x)^2}{x^3}$ (iii) $\frac{x^4}{x^2+1}$

(iv) $x\sqrt{x+2}$ (v) $(1+x)\sqrt{1-x}$

Solution.

(i) $\int \sqrt{x} - \frac{1}{\sqrt{x}}\ dx = \int \left(x^{1/2} - x^{-1/2}\right)dx$

$$= \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} = \frac{2x^{3/2}}{3} - 2x^{1/2} + C$$

(ii) $\int \frac{(1+x)^2}{x^3}\ dx = \int \left(\frac{1+2x+x^2}{x^3}\right) dx = \int \left(\frac{1}{x^3} + \frac{2}{x^2} + \frac{1}{x}\right) dx$

$$= \int x^{-3}\ dx + 2\int x^{-2}\ dx + \int \frac{1}{x} dx$$

$$= \frac{x^{-2}}{-2} + 2\frac{x^{-1}}{-1} + logx + C$$

$$= -\frac{1}{2x^2} - \frac{2}{x} + \log x + C.$$

(iii) $\int \frac{x^4}{x^2+1}\ dx = \int \left(\frac{(x^4-1)+1}{x^2+1}\right) dx$

$$= \int \frac{x^4-1}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \int (x^2-1)\ dx + \int \frac{1}{x^2+1} dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + C$$

(iv) $I = \int x\sqrt{x+2}\, dx$

$$= \int [(x+2)-2]\sqrt{x+2}\ dx$$

$$= \int (x+2)\sqrt{x+2}\ dx - \int 2\sqrt{x+2}\ dx$$

$$= \int (x+2)^{3/2}\ dx - 2\int (x+2)^{1/2}\ dx$$

$$= \frac{(x+2)^{5/2}}{5/2} - 2\frac{(x+2)^{3/2}}{3/2} + C$$

$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$

(v) I $= \int (1+x)\sqrt{1-x}\, dx$

$= \int [2-(1-x)]\sqrt{1-x}\, dx$

$= 2\int (1-x)^{1/2}\, dx - \int (1-x)^{3/2}\, dx$

$= \frac{2(1-x)^{3/2}}{-3/2} - \frac{(1-x)^{5/2}}{-5/2} + C$

$= -\frac{4}{5}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C.$

2.2.10. Example. Integrate $a^{3x+3}\, dx,\ a \neq -1$

Solution. I $= \int a^{3x+3} dx = \int a^{3x}.a^3 dx$

$= a^3 \int a^{3x} dx$

$= a^3 \int e^{3x\log a} dx$ (Since $e^{\log f(x)} = f(x)$

Therefore, $e^{\log a^{3x}} = a^{3x}$

Also $e^{\log\ a^{3x}} = e^{3xlog\ a}$

Therefore, $a^{3x} = e^{3xlog\ a}$

$= a^3 \int e^{(3\log a)x} dx$

$= a^3 \frac{e^{(3\log\ a)x}}{3\log a} + C$

$= a^3 \frac{e^{3x\log\ a}}{3\log a} + C = \frac{a^3 a^{3x}}{3\log a} + C = \frac{a^{3x+3}}{3\log a} + C.$

2.3. Integration by Substitution.

By substitution, many functions can be converted into smaller functions which can be integrated easily.

When we apply method of substitution for finding the value of $\int f(x)\ dx$ and if $x = f(t)$ where t is a new variable then $f(x)$ is converted into $F[f(t)]$ and also dy/dx.

Now $x = f(t)$

Therefore, $\frac{dx}{dt} = f'(t)$ or $dx = f'(t)\, dt$.

Two important forms of integrals :

(i) $\int \frac{f'(x)}{f(x)} dx = log|f(x)| + C$

(ii) $\int [f(x)]^n . f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1}$ $when\, n \neq -1$.

2.3.1. Example. Evaluate the following :

(i) $\int \frac{2x+9}{x^2+9x+10} dx$ (ii) $\int \frac{6x-8}{3x^2-8x+5} dx$

(iii) $\int 3x^2 . e^{x^3} dx$ (iv) $\int \frac{e^{1/x^2}}{x^3} dx$

(v) $\int \frac{\log x}{x} dx$ (vi) $\int \frac{1}{x \log_e x} dx$

(vii) $\int \frac{x^3}{\sqrt{1+x^3}} dx$ (viii) $\int \frac{1}{x+\sqrt{x}} dx$

Solution.

(i) $\mathrm{I} = \int \frac{2x+9}{x^2+9x+10} dx$

Put $x^2 + 9x + 10 = t$

Therefore, $2x + 9 = \frac{dx}{dt}$ or $(2x + 9)dx = dt$

Therefore, $\mathrm{I} = \int \frac{dt}{t} = \log|t| + C = \log|x^2 + 9x + 10| + C$

(ii) $\mathrm{I} = \int \frac{6x-8}{3x^2-8x+5} dx$

Put $3x^2 - 8x + 5 = t$

Therefore, $6x - 8 = \frac{dx}{dt}$ or $(6x - 8)dx = dt$

Therefore, $\mathrm{I} = \int \frac{dt}{t} = \log|t| + C = \log|3x^2 - 8x + 5| + C.$

(iii) $\mathrm{I} = \int 3x^2 . e^{x^3} dx$

Put $x^3 = t$

Therefore, $3x^2 = \frac{dx}{dt}$ or $3x^2 dx = dt$

Therefore, $\mathrm{I} = \int 3x^2 . e^{x^3} dx = \int e^t \, dt = e^t + C = e^{x^3} + C.$

(iv) Let $\frac{1}{x^2} = t$ or $x^{-2} = t$

Therefore, $-\frac{2}{x^3} dx = dt$ or $\frac{1}{x^3} dx = -\frac{1}{2} dt$

Thus, $\int \frac{e^{1/x^2}}{x^3} dx = \int e^{1/x^2} \frac{1}{x^3} dx = \int e^t \left(-\frac{1}{2}\right) dt$

$$= -\frac{1}{2} \int e^t \, dt = -\frac{1}{2} e^t + C = -\frac{1}{2} e^{1/x^2} + C.$$

(v) Let $\log x = t.$ So $\frac{1}{x} dx = dt$

Therefore,

$$\int \frac{\log x}{x} dx = \int \log x . \frac{1}{x} \, dx = \int t \, dt = \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C.$$

(vi) Let $\log_e x = t$, so $\frac{1}{x}dx = dt$

Therefore,

$$\int \frac{1}{x\log_e x}dx = \int \frac{1}{\log_e x}.\frac{1}{x}\,dx = \int \frac{1}{t}dt = \log_e t + C = \log_e(\log_e x) + C.$$

(vii) Let $1 + x^3 = t^2$ or $x^2dx = \frac{2}{3}t\,dt$

Therefore, $\int \frac{x^3}{\sqrt{(1+x^3)}}dx = \int \frac{x^3}{\sqrt{(1+x^3)}}.x^2dx = \int \frac{t^2-1}{t}.\frac{2}{3}t\,dt$

$$= \left(\frac{2}{3}\right)\int (t^2-1)dt = \frac{2}{3}\left(\frac{1}{3}t^3 - t\right) + C$$

$$= \frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + C$$

(viii) $\int \frac{1}{x+\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx$

Let $\sqrt{x} = t$, so $\frac{1}{2\sqrt{x}}dx = dt$ or $\frac{1}{\sqrt{x}}dx = 2dt$

Therefore, $\int \frac{1}{x+\sqrt{x}}dx = 2\int \frac{7}{\sqrt{x}(\sqrt{x}+1)} = \int \frac{1}{t+1}dx$

$$= 2\log(t+1) + C = 2\log\left(\sqrt{x}+1\right) + C.$$

2.3.2. Example. Integrate the following :

(i) $x\sqrt{x+2}$ (ii) $\frac{2+3x}{3+2x}$ (iii) $\frac{(x+1)(x+\log x)^2}{x}$ (iv) $\frac{1}{e^{x-1}}$

Solution.

(i) I = $\int x\sqrt{x+2}\,dx$

Putting $x + 2 = t$ implies $x = t - 2$

Therefore, $dx = dt$

$$\text{I} = \int (t-2)t^{1/2}\,dt = \int t^{3/2}\,dt - 2\int t^{1/2}\,dt$$

$$= \frac{t^{5/2}}{5/2} - 2\frac{t^{3/2}}{3/2} + C = \frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$$

(ii) I = $\int \frac{2+3x}{3+2x}\,dx$

Putting $3 - 2x = t$ implies $x = \frac{3-t}{2}$

Therefore, $dt = -2dx$ implies $dx = -\frac{dt}{2}$

$$\text{I} = -\frac{1}{2}\int \frac{2+3\left(\frac{3-t}{2}\right)}{t}dt = -\frac{1}{2}\int \frac{2+\frac{9}{2}-\frac{3}{2}t}{t}\frac{dt}{2}$$

$$= -\int \frac{dt}{t} - \frac{9}{4}\int \frac{dt}{t} + \frac{3}{4}\int dt$$

$$= -\log|t| - \frac{9}{4}\log|t| + \frac{4}{3}t + C$$

$$= -\log|3-2x| - \frac{9}{4}\log|3-2x| + \frac{3}{4}(3-2x) + C$$

$$= \frac{3}{4}(3-2x) - \log|3-2x| - \frac{9}{4}\log|3-2x| + C$$

(iii) $\mathrm{I} = \int \frac{(x+1)(x+\log x)^2}{x} dx$

Put $x + \log x = t,$ therefore $\left(1+\frac{1}{x}\right)dx = dt$ or $\left(\frac{x+1}{x}\right)dx = dt$

Thus $I = \int (x+\log x)^2 \left(\frac{x+1}{x}\right)dx = \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}\int (x+\log x)^3 + C.$

(iv) $\int \frac{1}{e^x - 1} dx = \int \frac{dt}{t(t-1)} = \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$

$$= \log(t-1) - logt + C$$

$$= log\left(\frac{t-1}{t}\right) = log\left(\frac{e^x-1}{e^x}\right) + C$$

2.4. Integral of the product of two functions.

If u and v be two functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

implies $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides w.r.t x, we get

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \text{.... (1)}$$

Let $u = f_1(x)$ and $\frac{dv}{dx} = f_2(x)$

Since $\frac{dv}{dx} = f_2(x)$, therefore $\int f_2(x)\,dx = v$

Hence (1) becomes

$$\int f_1(x) f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int [f_1'(x)\int f_2(x)dx]\,dx.$$

In words, this rule of integration by parts can be stated as :

Integral of the product of two functions

= First function . Integral of the second

-Integral of [diff. coeff. of the first . Integral of the second)

Integral of the product of two functions

In finding integrals by this method proper choice of 1st and 2nd function is essential. Although there is no fixed law for taking 1st and 2nd function and their choice is possible by practice, yet following rule is helpful in the choice of functions 1st and 2nd.

(i) If the two functions are of different types take that function as Ist which comes first in the word **ILATE.**

Where I, stands for Inverse circular function.

L, stands for Logarithmic function.

A, stands for Algebraic function.

T, stands for Trigonometrical function.

and E, stands for Exponential function.

(ii) If both the functions are trigonometrical take that function as 2nd whose integral is simpler.

(iii) If both the functions are algebraic take that function as 1st whose d.c. is simpler.

(iv) Unity may be taken as one of the functions.

(v) The formula of integration by parts can be applied more than once if necessary.

2.4.1. Example. Evaluate $\int x^n \log x\, dx$

Solution. Let $\mathrm{I} = \int x^n \log x\, dx = \int (\log x) x^n\, dx$

So $\mathrm{I} = (\log x)\frac{x^{n+1}}{n+1} - \int \frac{1}{x}\frac{x^{n+1}}{n+1} dx$

$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\int x^n\, dx$$

$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

2.4.2. Example. Evaluate $\int xe^x\, dx$

Solution. Let $\mathrm{I}= \int xe^x\, dx$

[Here x is algebraic function and e^x is exponential function and A occurs before T in ILATE, therefore, we take x as 1st and e^x as 2nd functions].

$$I = \int xe^x\, dx = x\int e^x\, dx - \int \left(\frac{d}{dx}(x)\int e^x\, dx\right) dx$$

$$= xe^x - \int 1.e^x\, dx = xe^x - e^x + C = e^x(x-1) + C.$$

2.4.3. Example. Evaluate $\int x^3 e^{-x}\, dx$

Solution. Let $\mathrm{I}= \int x^3 e^{-x}\, dx = x^3(-e^{-x}) - \int 3x^2(-e^{-x})dx$

$$= -x^3e^{-x} + 3\int x^2 e^{-x} dx$$

$$= -x^3e^{-x} + 3[x^2(-e^{-x}) - \int 2x(-e^{-x})dx]$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\int xe^{-x}dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6[x(-e^{-x}) - \int 1(-e^{-x})dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6e^{-x} + 6xe^{-x} + C$$

$$= -e^{-x}(x^3 + 3x^2 - 6x + 6) + C$$

2.4.4. Example. Integrate $x^3e^{x^2}$

Solution. $\mathrm{I} = \int x^3e^{x^2}\,dx$

Put $x^2 = t$, therefore, $2xdx = dt$ or $xdx = \frac{dt}{2}$

$$\mathrm{I} = \int x^3e^{x^2}\,dx = \int x^2e^{x^2}x\,dx = \int te^t\frac{dt}{2} = \frac{1}{2}\int te^t\,dt$$

$$= \frac{1}{2}[te^t - \int 1.e^t\,dt]$$

$$= \frac{1}{2}te^t - \frac{1}{2}e^t + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

2.4.5. Example. Evaluate $\int x^2e^{ax}\,dx$

Solution. Let $I = \int x^2e^{ax}\,dx$

$$= x^2\left(\frac{e^{ax}}{a}\right) - \int 2x\frac{e^{ax}}{a}dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[x\left(\frac{e^{ax}}{a}\right) - \int 1.\frac{e^{ax}}{a}dx\right]$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[x\frac{e^{ax}}{a} - \frac{1}{a}e^{ax}\right] + C$$

$$= e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^2}\right) + C.$$

2.4.6. Example. Evaluate $\int \log x\,dx$

Solution. Let I= $\int \log x\,dx = \int(\log x).1\,dx$

Integrating by parts, taking $log\ x$ as the 1st function

$$= \log x\,(x) - \int\frac{1}{x}.x\,dx = xlog\ x - \int 1\,dx$$

$$= x\log x - x + C = x(\log x - 1) + C.$$

2.4.7. Example. Evaluate $\int(\log x)^2.x\,dx$

Solution. Let $I = \int(\log x)^2.x\,dx$

$$= (\log x)^2.\frac{x^2}{2} - \int(2\log x).\frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}(\log x)^2 - \int(\log x).x\,dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left(\log x \frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\,dx\right)$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\left[(\log x)^2 - \log x + \frac{1}{2}\right] + C$$

2.4.8. Example. Evaluate $\int e^x(1+x)\log(xe^x)\,dx$

Solution. Let I= $\int e^x(1+x)\log(xe^x)\,dx$

Put $xe^x = t$, therefore, $e^x(1+x)dx = dt$

Therefore, $I = \int(\log t).1\,dt$

$$= \log t.(t) - \int \frac{1}{t}.t\,dt$$

$$= t\log t - \int 1.dt = t\log t - t + C$$

$$= t(\log t - 1) + C = (xe^x)[\log(xe^x) - \log e] + C$$

$$= (xe^x)\log\left(\frac{xe^x}{e}\right) + C$$

2.4.9. Example. Evaluate $\int \frac{\log x}{(x+1)^2}dx$

Solution. Let I= $\int \log x.\frac{1}{(x+1)^2}dx$

Now integrating by parts, taking $log\ x$ as first function

$$\mathrm{I} = \log x.\frac{-1}{1+x} - \int \frac{1}{x}.\frac{-1}{1+x}dx = -\frac{\log x}{1+x}dx = -\frac{\log x}{1+x} + \int \frac{1}{x(1+x)}dx$$

$$= -\frac{\log x}{1+x} + \int\left(\frac{1}{x} + \frac{1}{1+x}\right)dx$$

$$= -\frac{\log x}{1+x} + \log|x| - \log|1+x| + C$$

$$= -\frac{\log x}{1+x} + \log\left|\frac{x}{1+x}\right| + C.$$

2.5. Integration by partial fractions.

2.5.1. Rational Function. An expression of the form $\frac{f(x)}{\varphi(x)}$ where $f(x)$ and $\varphi(x)$ are rational integral algebraic functions or polynomials.

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$$

$$\varphi(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n.$$

Where m, n are positive integers and $a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n$ are constants is called a rational function or rational fraction. It is assumed that $f(x)$ and $\varphi(x)$ have no common factor.

e.g. $\frac{x+1}{x^3+x^2-6x}, \frac{x-1}{(x+1)(x^2+1)}$ are rational functions.

Such fractions can always be integrated by splitting the given fraction into partial fractions.

Note on Partial Fractions

1. **Proper rational algebraic fraction.** A proper rational algebraic fraction is a rational algebraic fraction in which the degree of the numerator is less than that of the denominator.

2. The degree of the numerator $f(x)$ must be less than the degree of denominator $\varphi(x)$ and if the degree of the numerator of a rational algebraic fraction is equal to or greater than, that of the denominator, we can divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

   Then

   Given fraction = a polynomial + a proper rational algebraic fraction.

   For example, consider a rational algebraic fraction.

   $$\frac{x^2}{(x-1)(x-2)} = \frac{x^2}{x^2-3x+2}$$

Hence the degree of the numerator is 3 and the degree of the denominator is 2. We divide numerator by denominator.

Therefore, $\frac{x^2}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)}$

Working rule.

(i) The degree of the numerator (x) must be less than the degree of denominator $\varphi(x)$ and if not so, then divide $f(x)$ by $\varphi(x)$ till the remainder of a lower degree than $\varphi(x)$.

(ii) Now break the denominator $\varphi(x)$ into linear and quadratic factors.

(iii) (a) Corresponding to non-repeated linear factor of $(x - a)$ type in the denominator $\varphi(x)$.. Put a partial fraction of the form $\frac{A}{x-\alpha}$.

Therefore, the partial fraction of $\frac{x^2}{(x+2)(x-4)(x-5)}$ are of the form $\frac{A}{x+2} + \frac{B}{x-4} + \frac{C}{x-5}$

(b) Corresponding to non-repeated quadratic factor $(ax^2 + bx + c)$ of $\varphi(x)$, partial fraction will be of the form $\frac{Ax+b}{ax^2+bx+c}$

For example, the partial fraction of

$$\frac{2x-3}{(x-1)(x-4)^2(x^2-5x+10)} = \frac{A}{(x-1)} + \frac{B}{x-4} + \frac{C}{(x-4)^2} + \frac{D}{x^2-5x+10}$$

(c) Corresponding to a repeated quadratic factor of the form $(ax^2 + b + c)^m$ in $\varphi(x)$, there corresponds m partial fractions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

Therefore the partial fractions of

$$\frac{3x-5}{(x+5)(x^2+7x+8)^2} = \frac{A}{x+5} + \frac{Bx+C}{x^2+7x+8} + \frac{Dx+E}{(x^2+7x+8)^2}$$

Thus we see that when we resolve the denominator $\varphi(x)$ into real factors, they can be of four types :

(a) Linear non-repeated.

(b) Linear repeated.

(c) Quadratic non-repeated.

(d) Quadratic repeated.

The proper fraction $\frac{f(x)}{\varphi(x)}$ is equal to the sum of partial fractions as suggested above. After this, multiply both sides by $\varphi(x)$. The relation, we get will be an identity. So the values of the constants of R.H.S. will be obtained by equating the coefficients of like powers of x, and then

solving the equation so obtained. Sometimes we can get the values of constants by some short

cut methods i.e., by giving certain values to x etc.

2.5.2. Example. Evaluate the following

(i) $\int \frac{3x+2}{(x-2)(2x+3)} dx$ (ii) $\int \frac{3x-1}{(2x+1)(3x+2)(6x-1)} dx$

Solution. (i) Let $\frac{3x+2}{(x-2)(2x+3)} = \frac{A}{x-2} + \frac{B}{2x+3}$

Multiplying both sides by $(x-2)\,(2x+3)$

$$3x + 2 = A(2x + 3) + B(x - 2)$$

Put $x = -\frac{3}{2}$, we have $B = \frac{5}{7}$

Put $x = 2$, we have $B = \frac{8}{7}$

Therefore, $\frac{3x+2}{(x-2)(2x+3)} = \frac{8}{7(x-2)} + \frac{5}{7(2x+3)}$

Thus, $\int \frac{3x+2}{(x-2)(2x+3)} dx = \int \frac{8}{7(x-2)} dx + \int \frac{5}{7(2x+3)} dx$

$$= \frac{8}{7}\log|x-2| + \frac{5}{7}\log|2x+3| + C$$

(ii) Let $\frac{3x-1}{(2x+1)(3x+2)(6x-1)} = \frac{A}{2x+1} + \frac{B}{3x+2} + \frac{C}{6x-5}$

Multiplying both sides by $(2x+1)\,(3x+2)\,(6x-5)$

$$(3x-1) = A(3x+2)(6x-5) + B(2x+1)(6x-5) + C(2x+1)(3x+2)$$

Put $x = -\frac{1}{2}$, we have $A = \frac{5}{8}$

Put $x = -\frac{2}{3}$, we have $B = -5$

Put $x = \frac{5}{6}$, we have $C = \frac{1}{8}$

Therefore, $\frac{3x-1}{(2x+1)(3x+2)(6x-1)} = \frac{5}{8(2x+1)} - \frac{1}{3x+2} + \frac{1}{8(6x-5)}$

Thus, $\int \frac{3x-1}{(2x+1)(3x+2)(6x-1)} dx = \int \frac{5}{8(2x+1)} dx - \int \frac{1}{3x+2} dx + \int \frac{1}{8(6x-5)} dx$

$$= \frac{5}{16}\log|2x+1| - \frac{1}{3}\log|3x+2| + \frac{1}{48}\log|6x-5| + C$$

2.5.3. Example. Evaluate

(i) $\int \frac{17x-2}{4x^2+7x-2} dx$ (ii) $\int \frac{dx}{x-x^3}$

Solution. (i) $\frac{17x-2}{4x^2+7x-2} = \frac{17x-2}{(x+2)(4x-1)} = \frac{A}{x+2} + \frac{B}{4x-1}$

$$17x - 2 = A(4x-1) + B(x+2)$$

Put $x = \frac{1}{4}$, we have $B = 1$

Put $x = -2$, we have $A = 4$

Therefore, $\frac{17x-2}{4x^2+7x-2} = \frac{4}{x+2} + \frac{1}{4x-1}$

Thus, $\int \frac{17x-2}{4x^2+7x-2} dx = \int \frac{4}{x+2} dx + \int \frac{1}{4x-1} dx$

$$= 4\log|x+2| + \frac{1}{4}\log|4x-1| + C.$$

(ii) $\int \frac{dx}{x-x^3} = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1-x)(x+x)}$

Let $\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$

Multiplying both sides by $x(1-x)(1+x)$, we get

$$1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$$

Putting $x = 0, 1$ and -1, we get $A = 1$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$

Putting these values of A, B and C, we get

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

Then,

$$\int \frac{dx}{x-x^2} = \int \left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right] dx$$

$$= \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + C$$

$$= \frac{1}{2}[2\log|x| - \log|1-x| - \log|1+x|] + C$$

$$= \frac{1}{2} \log \left| \frac{x^2}{1-x^2} \right| + C$$

2.5.4. Example. Evaluate (i) $\int \frac{dx}{1+3e^x+2e^{2x}}$ (ii) $\int \frac{dx}{6(\log x)^2+7\log x+2}$

Solution. (i) Put $e^x = t$, therefore $e^x dx = dt$

$$I = \int \frac{dt}{e^x(1+3t+2t^2)} = \int \frac{dt}{t(2t+1)(t+1)}$$

Now $$\frac{1}{t(2t+1)(t+1)} = \frac{1}{t} + \frac{1}{1+t} - \frac{4}{2t+1}$$

$$I = \int \frac{1}{t} dt + \int \frac{1}{1+t} dt - \int \frac{4}{2t+1} dt$$

$$= \log|t| + \log|1+t| - 2\log|2t+1| + C$$

$$= \log|e^x| + \log|e^x+1| - 2\log|2e^x+1| + C$$

$$= x + +\log|e^x+1| - 2\log|2e^x+1| + C$$

(ii) $\int \frac{dx}{6(\log x)^2+7\log x+2}$

Put $\log x = t$, then $\frac{1}{x} dx = dt$

$$I = \int \frac{dt}{6t^2+7t+2} = \int \frac{dt}{(2t+1)(3t+2)} = 2\int \frac{dt}{2t+1} - 3\int \frac{dt}{3t+2}$$

$$= \log|2t+1| - \frac{3}{2}\log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C = \log\left|\frac{2\log x+1}{3\log x+2}\right| + C$$

2.6. Definite Integral and Area.

Sometimes, in geometry and other branches of integral calculus, it becomes necessary to find the differences in two values (say a and b) of a variable x for integral values of function $f(x)$. This difference is called definite integral of $f(x)$ within limits a and b or b and a.

This definite integral is shown as follows :

$$\int_a^b f(x)\, dx$$

and is read as integration of $f(x)$ between limits a and b. As we know that if $\int f(x)\, dx = F(x)$

So $$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a),$$

where a and b are called lower and upper limits.

General Properties of Definite Integral

Property 1. $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Property 2. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Property 4. $\int_0^a f(x)\,dx = \int_0^b f(a-x)\,dx$

Property 5. $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function of x

$= 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function of x

Note. (i) $f(x)$ is called odd function if $f(-x) = -f(x)$

(ii) $f(x)$ is called even function if $f(-x) = f(x)$

Property 6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

2.6.1. Example. Find the values of

(i) $\int_0^1 x^2\,dx$ (ii) $\int_{-1}^2 (3x-1)(2x+1)\,dx$ (iii) $\int_2^3 \frac{dx}{x^2-1}$

(iv) $\int_0^2 \frac{e^x-e^{-x}}{5}dx$ (v) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$ (vi) $\int_0^1 \frac{dx}{[(ax+b)(1-x)]^2}$

Solution. (i) $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

(ii) $\int_{-1}^2 (3x-1)(2x+1)\,dx = \int_{-1}^2 (6x^2+x-1)\,dx$

$$= 6\left[\frac{x^3}{3}\right]_{-1}^2 + \left[\frac{x^2}{2}\right]_{-1}^2 + [x]_{-1}^2 = 16\frac{1}{2}$$

(iii) $\int_2^3 \frac{dx}{x^2-1} = \int_2^3 \frac{dx}{x^2-1^2}$

$$= \left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\left[\log\frac{2}{4} - \log\frac{1}{3}\right] = \frac{1}{2}\log\frac{3}{2}$$

(iv) $\int_0^2 \frac{e^x-e^{-x}}{5}dx = \frac{1}{5}\int_0^2 e^x - e^{-x}\,dx$

$$= \frac{1}{5}[e^x + e^{-x}]_0^2 = \frac{1}{5}\left(e - \frac{1}{e}\right)^2$$

(v) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int_0^1 \frac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}dx = \int_0^1 (\sqrt{x+1}-\sqrt{x})\,dx$

$$= \frac{2}{3}\left|(x+1)^{3/2} - x^{3/2}\right|_0^1 = \frac{4}{3}(\sqrt{2}-1).$$

(vi) $\int_0^1 \frac{dx}{[ax+b(1-x)]^2} = \int_0^1 \frac{dx}{[(a-b)x+b]^2} = \int_0^1 [(a-b)x+b]^{-2}\,dx$

$$= \left[\frac{[(a-b)x+b]^{-1}}{b-a}\right]_0^1 = \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{ab}$$

2.6.2. Example. If $\int_0^a 3x^2\,dx = 8$, find the value of a.

Solution. $\int_0^a 3x^2\,dx = 3\int_0^a x^2\,dx = 3\left[\frac{x^3}{3}\right]_0^a = a^3$

Since $\int_0^a 3x^2\,dx = 8$

Implies $a^3 = 8$ i.e. $a = 2$.

2.6.3. Example. Show that when $f(x)$ is of the form $a + bx + cx^2$, then

$$\int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

Solution. $f(x) = a + bx + cx^2$

$f(0) = a, \quad f\left(\frac{1}{2}\right) = a + \frac{1}{2}b + \frac{1}{4}c$

$f(1) = a + b + c$

RHS = $\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = a + \frac{b}{2} + \frac{c}{3}$

LHS = $\int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx = \left|ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right|_0^1 = a + \frac{b}{2} + \frac{c}{3}$

Hence. LHS = RHS.

2.6.4. Example. Evaluate the following definite integrals

(i) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx$ (ii) $\int_1^2 3x\sqrt{5-x^2}\,dx$ (iii) $\int_4^8 x\sqrt[3]{x-4}\,dx$

(iv) $\int_a^b \frac{\log x}{x}dx$ (v) $\int_0^2 \frac{dx}{4+x-x^2}$ (vi) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Solution. (i) $I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx$

Put $1 - x^2 = t$, then $-2xdx = dt$ or $xdx = -\frac{1}{2}dt$

When $x = 0, t = 1$

When $x = \frac{1}{2}, \ t = \frac{3}{4}$

Therefore, $I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int_1^{3/4} t^{-1/2}\,dt = -\frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right]_1^{3/4} = 1 - \frac{\sqrt{3}}{2}$

(ii) $I = \int_1^2 3x\sqrt{5-x^2}\,dx$

Put $5 - x^2 = t$, therefore $-2xdx = dt$ or $xdx = -\frac{1}{2}dt$

When $x = 1, t = 5 - 1 = 4$

When $x = 2, \ t = 5 - 4 = 1$

Therefore, $I = \int_1^2 3x\sqrt{5-x^2}\,dx = -\frac{3}{2}\int_4^1 t^{\frac{1}{2}}\,dt = -\frac{3}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^1$

$$= -(1-4^{\frac{3}{2}}) = -(1-8) = 7.$$

(iii) $I = \int_4^8 x\sqrt[3]{x-4}\,dx = \int_4^8 x(x-4)^{1/3}\ dx$

Put $x-4=t$, therefore, $dx = dt$

When $x=4,\ t=0$

When $x=8,\ t=4$

Therefore, $I = \int_0^4 (t+4)(t)^{1/3}\ dx = \int_0^4 t^{4/3}\,dt + 4\int_0^4 t^{1/3}\,dt$

$$= \frac{3}{7}\left|t^{7/3}\right|_0^4 + 4\times\frac{3}{4}\left|t^{4/3}\right|_0^4$$

$$= \frac{3}{7}(4)^{7/3} + 3(4)^{4/3} = \frac{132}{7}(4)^{1/3}.$$

(iv) $I = \int_a^b \frac{\log x}{x}\,dx$

Put $\log x = t$, therefore $\frac{1}{x}dx = dt$

When $x=a,\quad t=\log a$

When $x=b,\quad t=\log b$

Therefore, $I = \int_{\log a}^{\log b} t\,dt = \left|\frac{t^2}{2}\right|_{\log a}^{\log b} = \frac{1}{2}[(\log b)^2 - (\log a)^2]$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a) = \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$$

(v) $I = \int_0^2 \frac{dx}{4+x-x^2} = \int_0^2 \frac{dx}{4-(x^2-x)}$

$$= \int_0^2 \frac{dx}{4-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}} = \int_0^2 \frac{dx}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\sqrt{\frac{17}{2}}}\left[\log\left(\frac{x-\frac{1}{2}+\sqrt{\frac{17}{2}}}{-x+\frac{1}{2}+\sqrt{\frac{17}{2}}}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right) - \log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right\}$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})}\right) = \frac{1}{\sqrt{17}} \log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}} \times \frac{5+\sqrt{17}}{5+\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{21+5\sqrt{17}}{4}\right)$$

(vi) $I = \int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Put $x + 1 = t^2$, then $dx = 2tdt$

When $x = 8$, $t^2 = 9$, implies $t = 3$

When $x = 15$, $t^2 = 16$, implies $t = 4$

Therefore, $I = 2\int_3^4 \frac{dt}{t^2-2^2} = 2.\frac{1}{4}\left[\log\left|\frac{t-2}{t+2}\right|_3^4\right]$

$$= \frac{1}{2}\left(\log\left|\frac{2}{6}\right| - \log\frac{1}{5}\right) = \frac{1}{2}\left(\log\frac{1}{3} - \log\frac{1}{5}\right)$$

$$= \frac{1}{2}\log\frac{5}{3}$$

2.6.5. Example. Evaluate the following

(i) $\int_0^1 x^2 e^{2x}\, dx$ (ii) $\int_0^1 (x-2)(2x+3)e^x\, dx$

(iii) $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}} dx$ (iv) $\int_1^e \frac{e^x}{x}(1 + xlog\ x)$dx

Solution. (i) $I = \int_0^1 x^2 e^{2x}\, dx$

$$= \left|x^2\left(\frac{e^{2x}}{2}\right)\right|_0^1 - \int_0^1 2x\left(\frac{1}{2}e^{2x}\right) dx$$

$$= \frac{1}{2}(e^2 - 0) - \left[\left|\left(\frac{xe^{2x}}{2}\right)\right|_0^1 - \int_0^1\left(\frac{1}{2}e^{2x}\right) dx\right]$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{2}e^2 - \frac{1}{2}\int_0^1 e^{2x}\, dx\right] = \frac{1}{2}\int_0^1 e^{2x}\, dx$$

$$= \frac{1}{2}\left|\frac{1}{2}e^{2x}\right|_0^1 = \frac{1}{2}(e^2 - 1).$$

(ii) $I = \int_0^1 (x-2)(2x+3)e^x\, dx$

$$= \int_0^1 (2x^2 - x - 6)e^x\, dx$$

Integrating by parts, we get

$$= [(2x^2 - x - 6)e^x]_0^1 - \int_0^1 (4x-1)e^x\, dx$$

$$= (2 - 1 - 6)e - (-6) - \int_0^1 (4x-1)e^x\, dx$$

$= -5e + 6 - \left[\left|(4x-1)e^x\right|_0^1 - \int_0^1 4e^x\,dx\right]$

$= -5e + 6 - [(4-1)e - (-1) - 4|e^x|_0^1]$

$= -5e + 6[3e + 1 - 4(e-1)]$

$= 1 - 4e.$

(iii) $I = \int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx$

Integrating by parts taking $x^2 + x$ as first function and $\frac{1}{\sqrt{2x-1}}$ as the 2nd function.

$$I = \left|(x^2+x)\int \frac{dx}{\sqrt{2x+1}}\right|_2^4 - \int_2^4 \left\{(2x+1)\int \frac{dx}{\sqrt{2x+1}}\right\}dx$$

Now $\int \frac{dx}{\sqrt{2x+1}} = \frac{(2x+1)^{-\frac{1}{2}+1}}{2.\frac{1}{2}} = \sqrt{2x+1}$

Therefore, $I = \left|(x^2+x)\sqrt{2x+1}\right|_2^4 - \int_2^4 (2x+1)\sqrt{2x+1}\,dx$

$= \left(60 - 6\sqrt{5}\right) - \int_2^4 (2x+1)^{3/2}\,dx$

$= \left(60 - 6\sqrt{5}\right) - \left|\frac{(2x+1)^{5/2}}{5}\right|_2^4$

$= 60 - 6\sqrt{5} - \frac{1}{5}\left(9^{5/2} - 5^{5/2}\right) = 60 - 6\sqrt{5} - \frac{243}{5} + 5\sqrt{5}$

$= \frac{57}{2} - \sqrt{5}.$

(iv) $I = \int_1^e \frac{e^x}{x}(1 + x\log x)dx = \int_1^e e^x\left(\frac{1}{x} + \log x\right)dx$

$= \int e^x[f'(x) + f(x)]\,dx$ where $f(x) = \log x$

$= e^x f(x) = e^x \log x$

Therefore,

$$I = \int_1^e \frac{e^x}{x}(1 + x\log x)dx = [e^x \log x]_1^e = e^x \log e - e\log 1 = e^x$$

2.7. Definite Integral as area under the curve.

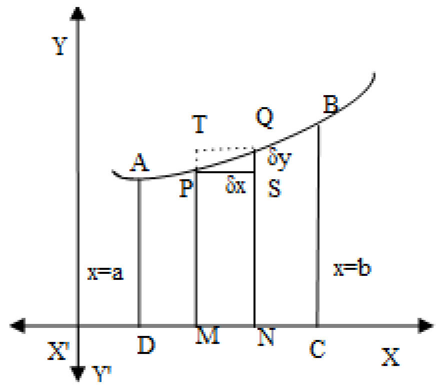
Let $f(x)$ be finite and continuous in $a \le x \le b$. Then area of the region bounded by x□axis, $y = f(x)$ and the ordinates at $x = a$ and $x = b$ is equal to $\int_a^b f(x)\,dx$.

Proof. Let AB be the curve $y = f(x)$ and $P(x, y)$ be any point on the curve such that $a \le x \le b$. Let DA and CB be the ordinates $x = a$ and $x = b$.

Take point $Q(x + \delta x, y + \delta y)$ near to the point $P(x, y)$. Draw PS and QT parallel to x-axis.

Clearly $PS = \delta x$ and $QS = \delta y$.

Let S represent the area bounded by the curve $y = f(x)$, x-axis and the ordinates AD $(x = a)$ and the variable ordinate PM.



Therefore, If δx is increment in x, then δS is increment in S.

It is clear from figure that δS is the area that lies between the rect. $PMNS$ and rect. $TQNM$.

Also area of rect. $PMNS = y.\delta x$ and area of rect. $TQNM = (y + \delta y)\,\delta x$

Therefore, $y\delta x < \delta S < (y + \delta y)\,\delta x$

Or $\quad y < \frac{\delta S}{\delta x} < y + \delta y$

When $Q \to P$, $\delta x \to 0$, $\delta y \to 0$

And $\quad \lim_{\delta x \to 0} \frac{\delta S}{\delta x} \to \frac{dS}{dx}$, we get

$\frac{dS}{dx} = y = f(x)$

Therefore, $\int_a^b f(x)\,dx = \int_a^b \frac{dS}{dx}.dx = \int_a^b ds = |S|_a^b = (S)_{x=b} - (S)_{x=a}$

But it is clear from the figure, when $x = a, S = 0$, because then PM and AD coincide and then $x = b$, S = area $ABCD$ = reqd. area.

Therefore,

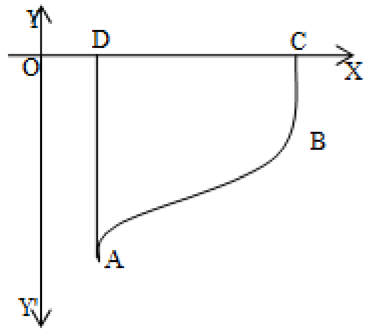
$\int_a^b f(x)\,dx =$ Area $ABCD$.

Thus the area bounded by the curve $y = f(x)$, the x axis and the ordinates $x = a$ and $x = b$ is $\int_a^b f\,(x)\,dx$.

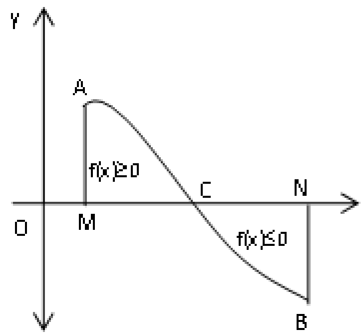
Remarks. In the figure given, we assumed that $f(x \geq 0)$ for all x in $a \leq x \leq b$. However, if

(i) $f(x) \leq 0$ for all x in $a \leq x \leq b$, then area bounded by x-axis, the curve $y = f(x)$ and the ordinate $x = a$ to $x = b$ is given by

$$= \int_a^b f\,(x)\,dx.$$



(ii) If $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $c \leq x \leq b$, then area bounded by $x = f(x)$, x-axis and the ordinates $x = a, x = b$, is

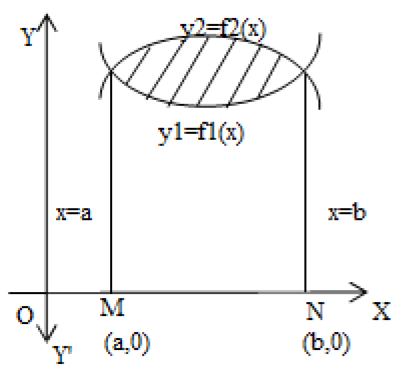


$$= \int_a^c f(x)\, dx + \int_c^b -f(x)\, dx$$

$$= \int_a^c f(x)\, dx - \int_c^b f(x)\, dx$$

(iii) The area of the region bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the ordinates $x = a$ and $x = b$ is given by

$$= \int_a^b f_2(x)\, dx - \int_a^b f_1(x)\, dx$$



where $f_2(x)$ is y_2 of upper curve and $f_1(x)$ is y_1 of lower curve i.e.,

Required area = $\int_a^b [f_2(x) - f_1(x)]dx = \int_a^b (y_2 - y_1)\, dx.$

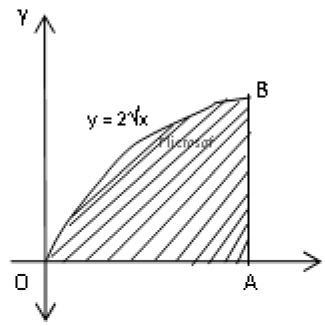
2.7.1. Example.

(a) Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

(b) Find the area under the curve $y = \sqrt{3x + 4}$ between $x = 0$ and $x = 4$.

Solution. (a) $y = 2\sqrt{x}$ implies $y^2 = 4x$

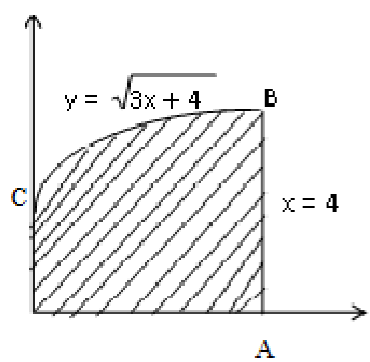
$y = 2\sqrt{x}$ is the upper part of the parabola $y^2 = 4x$. We have to find the area of the shaded region OAB.



Required area $= \int_0^1 y\,dx$

$$= \int_0^1 2\sqrt{x}\,dx = 2\left|\frac{x^{3/2}}{3/2}\right|_0^1 = \frac{4}{3} \text{ sq. units}$$

(b) $y = \sqrt{3x + 4}$, therefore, $y^2 = 3x + 4$. $y = \sqrt{3x + 4}$ is the upper part of the parabola $y^2 = 3x + 4$. We have to find the area of the shaded region.



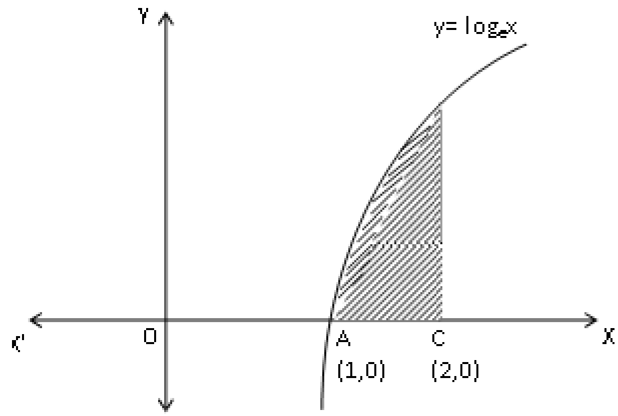
Required area OABC $= \int_0^4 y\,dx = \int_0^4 \sqrt{3x + 4}\,dx$

$$= \frac{2}{9}\left|(3x + 4)^{3/2}\right|_0^4 = \frac{112}{9} \text{ sq. units}$$

2.7.2. Example. Find the area bounded by $x = log_e\, x$, $y = 0$ and $x = 2$.

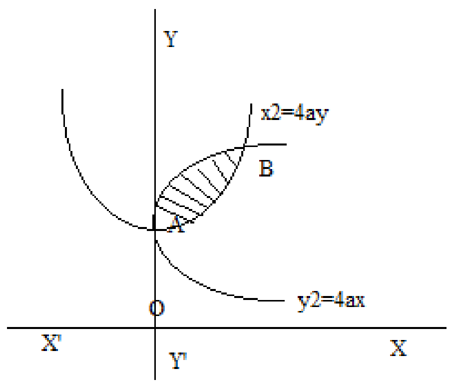
Solution. Required area ABC $= \int_1^2 y\,dx = \int_1^2 \log x\,dx$

$$= |x \log x - x|_1^2 = 2\log 2 - 1 = \log 4 - 1.$$



2.7.3. Example. Find the area included between two curves $y^2 = 4ax$ and $x^2 = 4\,ay$.

Solution. As shown in the figure, we have to find the area $OAPBO$.



Solving the given two equations simultaneously, we have

$$x^4 = 16a^2y^2 = 16a^2(4ax)$$

or $\quad x^4 = 64a^3x$ implies $x^4 - 64a^3x = 0,$

or $\quad x(x^3 - 64a^3) = 0$

or $\quad x = 0,\ x^3 = 64a^3$

Therefore, $x = 0$ at O and $x = 4a$ at B.

Now

Area $OAPBO$ = Area $OAPMO$ - Area $OBPMO$

$$= \int_0^{4a} y_1\,dx - \int_0^{4a} y_2\,dx = \int_0^{4a} 2a^{1/2}x^{1/2}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

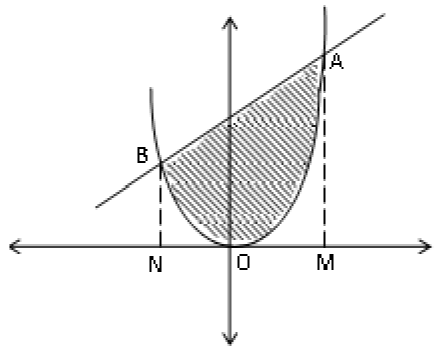
$$= 2a^{1/2}\int_0^{4a} x^{1/2}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2a^{1/2} \times \frac{2}{3}\left|x^{3/2}\right|_0^{4a} - \frac{1}{4a} \times \frac{1}{3}\left|x^3\right|_0^{4a}$$

$$= \frac{16}{3}a^2 \text{sq. units}$$

2.7.4. Example. Find the area cut-off from the parabola $4y = 3x^2$ by the straight line $2y = 3x + 12$.

Solution. Let the points of intersection of the parabola and the line be A and B as shown in the figure. Draw AM and BN perpendiculars to x-axis.



Now putting

$$y = \frac{3}{4}x^2 \text{ in } 2y = 3x + 12$$

We set $\frac{3}{2}x^2 = 3x + 12$

or $3x^2 - 6x + 24 = 0$

or $x^2 - 2x - 8 = 0$

or $x = 4,\ x = -2$

Then, $y = 12,\ y = 3.$

The co-ordinates of the point A are $(4, 12)$ and co-ordinates of B are $(-2, 3)$.

Now, Required area AOB

= Area of trapezium $BNMA$ -[Area BNO + Area OMA]

But area of trapezium

$= \frac{1}{2}$(sumof ||sides)×Height

$= \frac{1}{2} \times (12 + 3) \times 6 = 15 \times 3 = 45.$

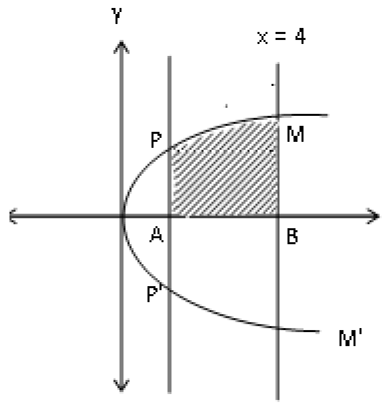
Area BNO + Area OMA $= \int_{-2}^{4} y\, dx$

$$= \frac{3}{4}\int_{-2}^{4} x^2\, dx = \frac{3}{4}\left|\frac{x^3}{3}\right|_{-2}^{4} = 18.$$

Hence required area = $45 - 18 = 27$ sq. units.

2.7.5. Example. Find the area bounded by the parabola $y^2 = 2x$ and the ordinates $x = 1$ and $x = 4$.

Solution. The equation of the parabola is $y^2 = 2x$ which is of the form $y^2 = 4ax$. The parabola is symmetrical about x-axis and opens towards right.



In the first quadrant $y \geq 0$.

Required Area = $PMM'P'$

$= 2$ area $ABMP$

$= 2\int_1^4 y\,dx = 2\int_1^4 \sqrt{2}x^{1/2}\,dx$

$= 2\sqrt{2}\left|\frac{x^{3/2}}{3/2}\right|_1^4 = \frac{28\sqrt{2}}{3}$ sq. units.

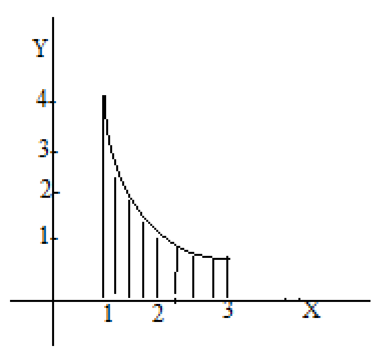
2.7.6. Example. Make a rough sketch of the graph of the function $y = \frac{4}{x^2}$ $(1 \leq x \leq 3)$, and find the area enclosed between the curve, the x-axis and the liens $x = 1$ and $x = 3$.

Solution. Given equation of the curve is

$$y = \frac{4}{x^2},\ (1 \leq x \leq 3)$$

Therefore, $y > 0$ i.e., the curve lies above the x-axis.

When $x = 1, 2, 3,$ then $y = 4, 1, 0.44$ respectively.

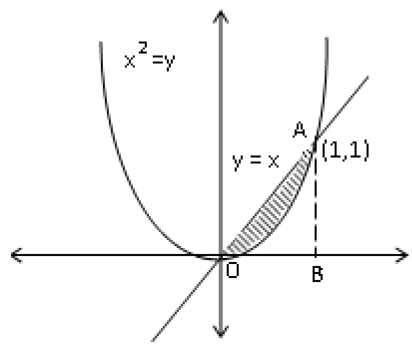


Required area $= \int_1^3 y\,dx$

$= \int_1^3 \frac{4}{x^2}\,dx = 4\left|-\frac{1}{x}\right|_1^3 = \frac{8}{3}$ sq. units.

2.7.7. Example. Find the area of the region $\{(x, y) : x^2 \leq y \leq x\}$.

Solution. Let us first sketch the region whose area is to be found out. The required area is the area included between the curves $x^2 = y$ and $y = x$.



Solving these two equations simultaneously, we have

$x^2 = x$ implies $x^2 - x = 0$

or $x(x-1) = 0$

or $x = 0, x = 1$

When $x = 0,\ y = 0$ and $x = 1,\ y = 1$.

Therefore, these two curves intersect each other at two points $O(0,0)$ and $A(1,1)$.

$$\text{Required area} = \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{6} \text{ sq. units.}$$

2.7.8. Example. Find the area of the region $\{(x,y) : x^2 \le y \le |x|\}$.

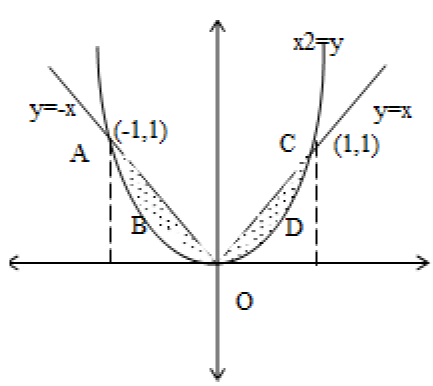
Solution. Let us first sketch the region whose area is to be found out.

The required area is the area included between the curves $x^2 = y$ and $y = |x|$.

The graph of $x^2 = y$ is a parabola with vertex (0, 0) and axis y-axis as shown in figure.

The graph of $y = |x|$ is the union of lines $y = x, x \ge 0$ and $y = -x, x \le 0$.

The required region is the shaded region.



Therefore, the required area = Area OAB+Area OCD

$$=2 \text{ Area OCD}$$

$$= 2\int_0^1 x\,dx - 2\int_0^1 x^2\,dx$$

$$= 2\left|\frac{x^2}{2}\right|_0^1 - 2\left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3} \text{ sq. units.}$$

2.7.9. Example. Using integration find the area of the triangular region whose sides have the Equation

$$y = 2x + 1 \qquad \ldots(1)$$

$$y = 3x + 1 \qquad \ldots(2)$$

and $\quad x = 4 \qquad \ldots(3)$

Solution. Solving (1) and (3), we get $x = 4, y = 2 \times 4 + 1 = 9.$

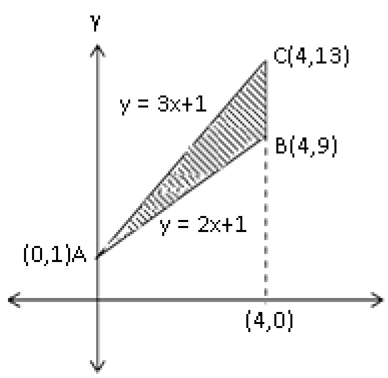
Therefore, $(4, 9)$ is the point of intersection of lines (1) and (3).

Solving (1) and (2), we get $x = 0,\ y = 1.$

Therefore, $(0, 1)$ is the point of intersection of lines (1) and (2).

Solving (2) and (3), we get $x = 4,\ y = 3 \times 4 + 1 = 13$.

Therefore, $(4, 13)$ is the point of intersection of lines (2) and (3)



Required area ABC = $\int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$

$$= \left|\frac{3x^2}{2} + x\right|_0^4 - \left|\frac{2x^2}{2} + x\right|_0^4 = 8 \text{ sq. units.}$$

2.8. Learning Curve.

Learning curve is a technique with the help of which we can estimate the cost and time of production process of a product. With passage of time, the production process becomes increasingly mature and reaches a steady state. It so happens because with gain in experience with time, time taken to produce one unit of a product steadily decreases and in the last attains a stable value.

The general form of the learning curve is given by

$$y = f(x) = ax^{-b}$$

where y is the average time taken to produce one unit, and x is the number of units produced, a and b are the constants.

a is defined as the time taken for producing the first unit ($x = I$) and b is calculated by using the formula

$$b = \frac{\log(learning\ \ rate)}{\log 2}$$

If the learning curve is known, then total time (labour hours) required to produce units numbered from a to b is given by

$$L = \int_a^b f(x)\,dx = \int_a^b A.x^a\,dx \qquad \text{(another form of learning curve)}$$

2.8.1. Example. The first batch of 10 dolls is produced in 30 hours. Determine the time taken to produce next 10 dolls and again next 20 dolls, assuming a 60% learning rate. Estimate the time taken to produce first unit.

New Time taken to produce one batch

= Previous time taken to produce one batch × learning rate

No. of dolls	**Total time (hours)**	**Total increase in time**	**Average time**
0	0	-	-
10	30	30	3
20	$20\left(\frac{30\times 60}{100}\right) = 36$	6	1.8
40	$20\left(\frac{36\times 60}{100}\right) = 43.2$	7.2	1.08

Now, $\beta = -\frac{\log(0.6)}{\log 2} = -0.7369$

When $x = 10, y = 3$, then $3 = a.10^{-0.7369}$

Solving the equation, we get $A = 16.38$ hours.

2.8.2. Example. Because of learning experience, there is a reduction in labour requirement in a firm. After producing 36 units, the firm has the learning curve $f(x) = 1000\,x^{-1/2}$. Find the labour hours required to produce the next 28 units.

Solution L = $\int_{36}^{64} 1000x^{-0.5}\,dx$

$= 1000[2x^{1/2}]_{36}^{64} = 2000[8-6] = 4000$ hours.

2.8.3. Example. A firm's learning curve after producing 100 units is given by $f(x) = 2400\, x^{-0.5}$ which is the rate of labour hours required to produce the x^{th} unit. Find the hours needed to produce an additional 800 units.

Solution. Labour hours required

$$L = \int_{100}^{900} f(x)\, dx$$

$$= \int_{100}^{900} 2400x^{-0.5}\, dx = 2400\left[\frac{x^{1/2}}{1/2}\right]_{100}^{900} = 4800[30 - 10] = 96000 \text{ hours.}$$

2.9. Consumer and Producer Surplus.

Consumer surplus is the difference between the price that a consumer is willing to pay and the actual price he pays for a commodity. The degree of satisfaction derived from a commodity is a subjective matter.

If DD_1 is the market demand curve then demand x_0 corresponds to the price p_0. The consumer surplus is given by DD_1p_0.

$$DD_1p_0 = \text{Area } DD_1x_00 - \varphi_0D_1x_0O$$

$$= \int_0^{x_0} f(x)\, dx - p_0x_0$$

where $f(x)$ is the demand function.

It is assumed that the area is defined at $x = 0$ and that the satisfaction is measurable in terms of price for all consumers. In other words, we assume that utility function is same for all consumers and marginal utility of money is constant.

2.9.1. Example. Find the consumer surplus if the demand function is $p = 25 - 2x$ and the surplus function is $4p = 10 + x$.

Solution. First find the equilibrium price p_0 and equilibrium demand, x_0 by solving the above two equations simultaneously, we have

$$x_0 = 10 \text{ and } p_0 = 5$$

Now consumer surplus $= \int_0^{x_0} f(x)\, dx - p_0x_0$

$$= \int_0^{10} (25 - 2x)\, dx - 5 \times 10$$

$$= [25x - x^2]_0^{10} - 50 = 100.$$

2.10. Producer Surplus

Producer surplus is the difference in the prices a producer expects to get and the price which he actually gets for a commodity.

If SS_1 is the market supply curve and if x_0 is the supply at the market price p_0, the producer surplus is the area PS.

$$PS = \text{Area } SS_1P_0 = p_0x_0 - \int_0^x g(x)\,dx$$

where $g(x)$ is the supply function.

2.10.1. Example. Find the producer surplus for the supply function $p^2 - x = 9$ when $x_0 = 7$

Solution. We are given $p^2 - x = 9$ or $p_0^2 - x_0 = 9$

Also given $x_0 = 7$

Therefore, $p_0 = 7$

Thus,

$$\begin{aligned} PS &= p_0x_0 - \int_0^x g(x)\,dx \\ &= 4 \times 7 - \int_0^7 (x+9)^{1/2}\,dx \\ &= 28 - \left[\frac{2}{3}(x+9)^{3/2}\right]_0^7 = \frac{10}{3}. \end{aligned}$$

2.11. Leontief Input-Output Model.

Consider a model consisting of n production units and each unit produces only one kind of product. Each unit in the model uses the output of these n units as input. Also some part of output of each unit is used by other consumers, we shall call those parts as final demand of the unit. The sum of all the outputs of a particular unit is known as total output of that unit. Now, we have to determine the new total output of a unit if the final demand changes assuming that the resources of the model does not change. Here comes the role of Leontief input-output model. We illustrate the process for three production units.

1. Avaiblable data. Let P_1, P_2, P_3 be three production units and x_{ij} denote the part of output of the unit P_i used as input by the units P_j. Let b_i denotes the final demand of unit P_i and X_i denotes the total output of unit P_i. This data can be represented as:

Production Unit	Receiving unit			Final demand	Total output
	P₁	**P₂**	**P₃**		
P_1	x_{11}	x_{12}	x_{13}	b_1	X_1
P_2	x_{21}	x_{22}	x_{23}	b_2	X_2
P_3	x_{31}	x_{32}	x_{33}	b_3	X_3

2. Construction of Technology matrix. The ratio $\dfrac{x_{ij}}{X_j}$ is denoted by a_{ij} and is known as input-output coefficients or technical coefficients. For example,

$$a_{11} = \frac{x_{11}}{X_1},\ a_{12} = \frac{x_{12}}{X_2},\ a_{13} = \frac{x_{13}}{X_3}$$

Then the matrix A of all these input-output coefficients is called Technology matrix or matrix of technical coefficients. Thus the technical matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3. Simon-Hawkins Conditions. The conditions for the system to be viable are:

(i) The elements on the principal diagonal of Leontief matrix are all positive i.e., $1-a_{11}, 1-a_{22},...$ are all positive.

(ii) The determinant of Leontief matrix i.e., $|I-A|$ is positive.

If these two conditions are not satisfied, then there is no solution of the above system. These conditions are known as (Simon-Hawkins conditions for viability of system)

2.11.1. Example. For a two unit economy with production units X and Y, the inter unital demand and final demand are as follows :

Production Unit	Receiving unit P1	Receiving unit P2	Final demand	Total output
P_1	30	40	50	120
P_2	20	10	30	60

(i) Find the technical coefficients.

(ii) Find the matrix of technical coefficients.

(iii) Find the Leontief matrix

(iv) Verify Simon-Hawkins conditions for viability of the system.

Solution. The given table is

Production / Unit	Receiving unit P1	Receiving unit P2	Final demand	Total output
P1	$x_{11} = 30$	$x_{21} = 40$	$b_1 = 50$	$X_1 = 120$
P2	$x_{12} = 20$	$x_{22} = 10$	$b_2 = 30$	$X_2 = 60$

(i) Here n = 2, technical co-efficients are $a_{11}\ a_{21}$, a_{12} , a_{22}

Thus
$$a_{11} = \frac{x_{11}}{X_1} = \frac{30}{120} = \frac{1}{4}$$
$$a_{21} = \frac{x_{21}}{X_1} = \frac{20}{120} = \frac{1}{6}$$
$$a_{12} = \frac{x_{12}}{X_2} = \frac{40}{60} = \frac{2}{3}$$
$$a_{22} = \frac{x_{22}}{X_2} = \frac{10}{60} = \frac{1}{6}$$

(ii) Matrix of technical coefficients $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

(iii) Leontief Matrix $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$

(iv) Elements on principal diagonal of Leontief matrix are $\frac{3}{4}$ and $\frac{5}{6}$ which are positive . Also, $|I - A|$

$$= \begin{vmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{vmatrix} = \frac{37}{72} > 0$$

Hence Simon-Hawkins conditions are verified.

2.12. Check Your Progress.

1. Evaluate (i) $\int (4x^3 + 3x^2 - 2x + 5)dx$ (ii) $\int \left(\sqrt{x} - \frac{1}{2}x + \frac{2}{\sqrt{x}}\right) dx$

(iii) $\int \left(\frac{x^4+1}{x^2}\right) dx$ (iv) $\int \left(x - \frac{1}{x}\right)^3 dx$

(v) $\int \left(2^x + \frac{1}{2}e^{-x} + \frac{4}{x} - \frac{1}{\sqrt[3]{x}}\right) dx$ (vi) $\int \frac{x}{x-3} dx$

(vii) $\int \left(e^{a \log x} + e^{x \log a}\right)dx$ (viii) $\int \frac{1}{\sqrt{5x+3} - \sqrt{5x+2}} dx$

(ix) $\int \left(e^{3x} - 2e^x + \frac{1}{x}\right) dx$ (x) $\int \frac{(x^3+1)(x-2)}{x^2-x-2} dx$

(xi) $\int \frac{(a^x+b^x)^2}{a^x b^x} dx$ (xii) $\int \frac{1}{\sqrt{x+1} + \sqrt{x-1}} dx$

2. Evaluation (i) $\int \frac{3x+5}{(3x^2+10x+2)^{2/3}} dx$ (ii) $\int \frac{\sqrt{2+\log x}}{x} dx$

(iii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ (iv) $\int \frac{dx}{x^2-a^2}$

(v) $\int x(x^2 + 4)^5\, dx$ (vi) $\int \frac{8x^2}{(x^3+2)^3} dx$

(vii) $\int \frac{x^3}{(x^2+1)^3} dx$ (viii) $\int \frac{x+2}{\sqrt{x^2+4x+5}} dx$

(ix) $\int \frac{(x+1)(x+\log x)^3}{3x} dx$ (x) $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e} dx$

(xi) $\int \frac{1}{(1+e^x)(1+e^{-x})} dx$ (xii) $\int (x+1)2^{x^2+2x}\, dx$

3. Evaluate (i) $\int x^2 e^{3x}\, dx$ (ii) $\int x^n \log x\, dx$

(iii) $\int \frac{xe^x}{(x+1)^2} dx$ (iv) $\int \log x\, dx$

(v) $\int \sqrt{4x^2-9}\, dx$ (vi) $\int \frac{dx}{(x+1)\sqrt{x+2}}$

(vii) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$ (viii) $\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$

(ix) $\int x \log(1+x)\, dx$ (x) $\int x^3 a^{x^2} dx$

4. Evaluate. (i) $\int \frac{x}{(x-1)(x-2)} dx$ (ii) $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

(iii) $\int \frac{3x+5}{x^4-x^3-x^2+1} dx$ (iv) $\int \frac{x^2+1}{(2x+1)(x+1)(x-1)} dx$

(v) $\int \frac{26x+6}{8-10x-3x^2} dx$ (vi) $\int \frac{2x^3-3x^2-9x+1}{2x^2-x-10} dx$

(vii) $\int \frac{dx}{(x+1)^2(x^2+1)} dx$ (viii) $\int \frac{dx}{(e^x-1)^2}$

(ix) $\int \frac{x^2+x+1}{(x-3)^3} dx$ (x) $\int \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)} dx$

5. Evaluate the following:

(i) $\int_2^4 (3x-2)^2\, dx$ (ii) $\int_6^{10} \frac{1}{x+2} dx$ (iii) $\int_3^{11} \sqrt{2x+3}\, dx$

(iv) $\int_0^2 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{2-x}} dx$ (v) $\int_0^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$ (vi) $\int_0^1 \frac{3x^3-4x^2+1}{\sqrt{x}} dx$

6. Evaluate the following :

(i) $\int_0^1 \frac{x^5}{1+x^6} dx$ (ii) $\int_1^2 x\sqrt{3x-2}\, dx$ (iii) $\int_2^4 \frac{6x^2-1}{\sqrt{2x^3-x-2}} dx$

(iv) $\int_1^2 \frac{(\log x)^2}{x} dx$

7. Evaluate the following :

(i) $\int_0^1 xe^x\, dx$ (ii) $\int_0^1 x\log\left(1+\frac{x}{2}\right) dx$ (iii) $\int_0^1 x^2 e^x\, dx$

(iv) $\int_a^b \frac{\log x}{x^2} dx$

8. Find the area of the region included between the parabola $y = \frac{3}{4}x^2$ and the line $3x - 2y + 12 = 0$.

9. Find the area bounded by the curve $y = x^2$ and the line $y = x$.

10. Make a rough sketch of the graph of the function $y = 9 - x^2,\ 0 \le x \le 3$ and determine the area enclosed between the curve and the axis.
11. Using integration, find the area of the region bounded by the triangle whose vertices are $(1,0), (2,2)$ and $(3,1)$.
12. Find the area of the region bounded by $y = -1,\ y = 2,\ x = y^2,\ x = 0$.
13. Find the area between the parabola $y^2 = x$ and the line $x = 4$.
14. Find the area bounded by the curve $y = x^2 - 4$ and the lines $y = 0$ and $y = 5$.
15. Find the area of the region enclosed between the curve $y = x^2 + 1$ and the line $y = 2x + 1$.
16. Find the area bounded by the curve $x = at^2,\ y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.
17. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and chord $y = mx$.

Answers.

1. (i) $x^4 + x^3 - x^2 + 5x + C$ (ii) $\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4\sqrt{x} + C$

(iii) $\frac{x^3}{3} - \frac{1}{x} + C$ (iv) $\frac{x^4}{4} - \frac{3}{2}x^2 + 3\log x + \frac{1}{2x^2} + C$

(v) $\frac{2^x}{log\, 2} - \frac{1}{2}e^{-x} + 4\log x - \frac{3}{2}x^{2/3} + C$ (vi) $x + \log|x-3| + C$

(vii) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + C$ (viii) $\frac{2}{15}\left[(5x+3)^{\frac{3}{2}} - (5x+2)^{\frac{3}{2}}\right] + C$

(ix) $\frac{e^{3x}}{3} - 2e^x + \log|x| + C$ (x) $\frac{x^3}{3} - \frac{x^2}{2} + x + C$

(xi) $\frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}} + 2x + \frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}} + C$ (xii) $\frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C$

2. (i) $\frac{3}{2}(3x^2 + 10x + 2)^{\frac{1}{3}} + C$ (ii) $\frac{2}{3}(2 + \log x)^{\frac{3}{2}} + C$ (iii) $\log|e^x + e^{-x}| + C$

(iv) $\frac{1}{2a} log\frac{|x-a|}{|x+a|} + C$ (v) $\frac{1}{12}(x^2+4)^6$ (vi) $\frac{4}{3(x^2+2)^2}$

(vii) $-\frac{1}{4}\frac{2x^2+1}{(x^2+1)^2}$ (viii) $\frac{1}{15}(1+x^6)^{5/2} + C$ (ix) $\frac{1}{12}(x + \log x)^4 + C$

(x) $\frac{1}{e}\log|e^x + x^e| + C$ (xi) $-\frac{1}{1+e^x} + C$ (xii) $\frac{2^{x^2+2x}}{2\log 2} + C$

3. (i) $\frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27}e^{3x}$ (ii) $\log x\ \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}$ (iii) $\frac{1}{x+1}e^x$

(iv) $x(\log x)^2 - 2xlog\, x + 2x$ (v) $\frac{\sqrt{4x^2-9}}{2} - \frac{9}{4}\log|2x + \sqrt{4x^2-9}| + C$

(vi) $\log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$ (vii) $1 - \log\left|\frac{1}{x+1} - \frac{1}{2} + \frac{\sqrt{x^2+x+1}}{x+1}\right| + C$

(viii) $-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2x}+\sqrt{x^2+1}}{\sqrt{2x}-\sqrt{x^2+1}}\right| + C$ (ix) $\frac{1}{2}(x^2-1)\log(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + C$

(x) $\frac{x^2 a^{x^2}}{2\log a} - \frac{a^{x^2}}{2(\log a)^2} + C$

4. (i) $-\log|x-1| + 2\log|x-2| + C$ (ii) $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$

(iii) $\frac{1}{2}log\ \left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$ (iv) $-\frac{5}{6}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + C$

(v) $-\frac{5}{3}\log|3x-2| - 7\log|x+4| + C$ (vi) $\frac{x^2}{2} - x + \log\left|\frac{x+2}{2x-5}\right| + C$

(vii) $\frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + C$

(viii) $\log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + C$ (ix) $\log|x-3| - \frac{7}{x-3} - \frac{13}{2(x-3)^2} + C$

(x) $a^3 + ab + clog\ |x-a| + \frac{ab^2+b^2+c}{(b-a)(b-c)}\log|x-b| + \frac{c(ac+b+1)}{(c-a)(c-b)}\log|x-c| + k$

5. (i) 104 (ii) $\log\frac{3}{2}$ (iii) $\frac{98}{3}$ (iv) 1 (v) $\frac{1}{\sqrt{3}}$ (vi) $-\frac{52}{15}$

6. (i) $\frac{1}{6}\log 2$ (ii) $\frac{326}{135}$ (iii) $2(\sqrt{122} - \sqrt{12})$ (iv) $\frac{1}{3}(\log 2)^3$

7. (i) 1 (ii) $\frac{3}{4} + \frac{2}{3}\log\frac{2}{3}$ (iii) e^{-2} (iv) $\frac{\log a+1}{\log a} - \frac{\log b+1}{\log b}$

8. 27 9. $\frac{1}{6}$ 10. 18 11. $\frac{3}{2}$

12. $\frac{15}{4}$ 13. $\frac{32}{3}$ 14.. $\frac{76}{3}$ 15. $\frac{4}{3}$

16. $\frac{56a^2}{3}$ 17. $\frac{8a^2}{3m^2}$

2.13. Summary. In this chapter, we derived methods to find the integration of various functions by using various methods. Also, Leontiff input-output model is discussed.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.